

QuDot Nets: Quantum Computers and Bayesian Networks

QuDot

1 Introduction

Quantum computers can solve some computational problems exponentially faster than classical computers which has applications in the fields of cryptography [1], machine learning [2] and physics simulations [3]. These high profile applications of quantum computation have lead to a race to build new quantum computing hardware, but does quantum computation require new hardware? Quantum computation is a model of computation, much like Turing Machines are a model of computation. It is widely accepted but not proven that new hardware is needed to efficiently implement quantum computation and reach quantum supremacy. Quantum supremacy is defined to be a computation possible on a quantum computer but impossible on the largest supercomputer in existence. Probing the quantum supremacy boundary via classical simulation will be an invaluable tool for industrial application of quantum computation, similarly to how classical integrated circuits are simulated before they are implemented. Thus, tools that can simulate large quantum computers at the boundary of quantum supremacy will be in every quantum engineer's toolbox. Analyzing algorithms in the quantum supremacy region may lead to a better understanding of the difference between classical and quantum computation.

We present a new graph theoretic implementation of the quantum computation model based on Bayesian Networks: QuDot Nets (QuNets). QuNets implement n -qubit states as nodes (or dots) on a directed, acyclic, weighted multi-graph (multi-digraph). The nodes are labeled 0 or 1 representing the possible values of a single qubit. An edge between nodes represents a tensor product and the edge weight represents a probability amplitude to observe the child node given the parent node. QuNets are able to represent non-entangled n -qubit states using a linear amount of resources as opposed to the 2^n numbers required by a vector implementation. Single qubit quantum gates are implemented as transformations of edge weights thus allowing QuNets to implement single qubit gates, measurements and semi-quantum control gates in linear time as opposed to requiring $2^n \times 2^n$ matrix multiplications. Coherent two qubit control gates can be implemented by allowing multiple labeled edges to connect nodes making the QuDot Net a multi-digraph. The number of edges can still grow exponentially for coherent control gates, however, it remains an open problem to determine if duplicate edges can be efficiently merged.

2 Representation and Measurement

QuNets are able to represent an n qubit quantum state $|q\rangle$ if the quantum state starts in the ground state $|000\dots 0\rangle$ and can be evolved to the desired state $|q\rangle$ using quantum gates defined as edge transformations on QuNets. QuNets represent a qubit as two graphical nodes side by side in a layer called a qubit layer. The number of qubit layers represents the total number of qubits, the top most qubit layer is the first qubit and the n -th qubit layer is the n -th qubit. The first qubit layer is accessed by a special node called the root node. The left node of a qubit layer is labeled the 0 node and the right node of the qubit layer is labeled the 1 node. The two nodes in each qubit layer are called siblings. Directed, weighted edges go from a node in one qubit layer to a node in another qubit layer and represents a tensor product between those qubits. The weights of the directed edges are complex numbers and correspond to the probability amplitude to observe the child node given the parent node. There are $v = 2q + 1$ number of nodes to represent q qubits and $e = 2v - 4$ edges. Therefore, the QuNet representation scales linearly in the number of qubits q for qubit systems that are generated by starting in the ground state and evolved by using single qubit quantum gates defined on QuNets (H , $R(k)$, M and SWAP) and semi-quantum control gates. It remains to be seen if we can extend the linear (or polynomial) efficiency to coherent control gates.

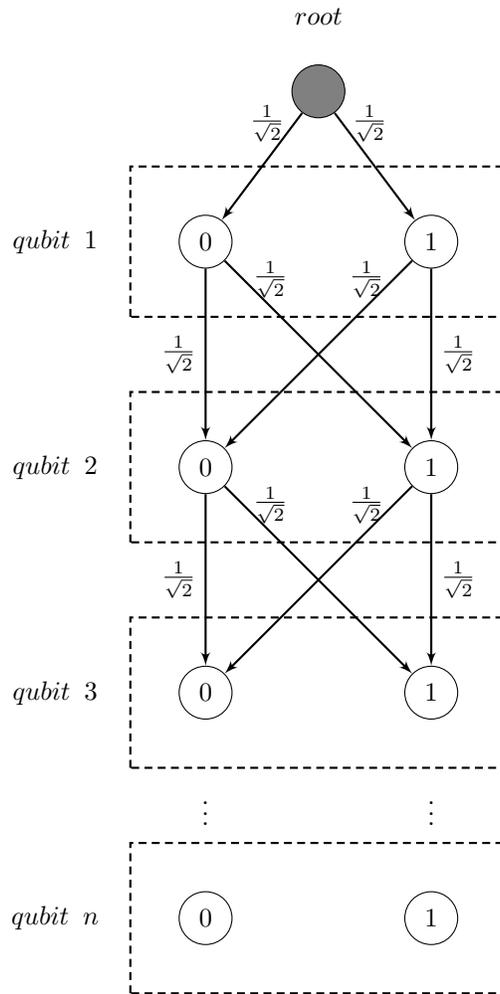


Figure 1: shows graphical QuDot Net representation of an n -qubit superposition

Graph traversals of a QuNet correspond to quantum mechanical measurements in the computational basis. A possible traversal of the QuNet is a possible measurement outcome of the qubit state and the edge weights of the specific traversal multiplied together corresponds to the probability amplitude of that measurement. All possible traversals of the QuNet with a nonzero probability amplitude correspond to all possible measurement outcomes of the qubit state. A single traversal takes linear time and corresponds to a measurement in quantum mechanics. Obtaining all possible probability amplitudes requires the enumeration of all 2^n possible traversals of the QuNet. This corresponds to the fact that Nature “hides” the probability amplitudes, therefore, QuNets are consistent with the framework of quantum mechanics.

Consider $|q\rangle = \frac{1}{\sqrt{8}} (|000\rangle + |001\rangle + |010\rangle + |011\rangle + |100\rangle + |101\rangle + |110\rangle + |111\rangle)$

There are eight possible measurement outcomes of $|q\rangle$ corresponding to the possible traversals of the QuDot Net in figure 2:

1. $|000\rangle$ with probability amplitude $\frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}} = \frac{1}{2\sqrt{2}}$
2. $|001\rangle$ with probability amplitude $\frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}} = \frac{1}{2\sqrt{2}}$

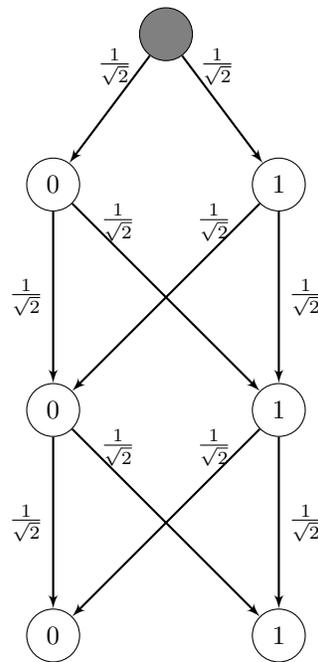


Figure 2: shows QuDot Net of $|q\rangle$

3. $|010\rangle$ with probability amplitude $\frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}} = \frac{1}{2\sqrt{2}}$
4. $|011\rangle$ with probability amplitude $\frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}} = \frac{1}{2\sqrt{2}}$
5. $|100\rangle$ with probability amplitude $\frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}} = \frac{1}{2\sqrt{2}}$
6. $|101\rangle$ with probability amplitude $\frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}} = \frac{1}{2\sqrt{2}}$
7. $|110\rangle$ with probability amplitude $\frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}} = \frac{1}{2\sqrt{2}}$
8. $|111\rangle$ with probability amplitude $\frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}} = \frac{1}{2\sqrt{2}}$

Consider $|g\rangle = |000\rangle$

There is only one possible measurement outcome of $|g\rangle$ corresponding to the one nonzero traversal of the QuNet in figure 3:

1. $|000\rangle$ with probability amplitude $1 \times 1 \times 1 = 1$

QuDot Net Traversal Algorithm: begin on root, while a node has child edges let p_0, p_1 be the edge weights of the child edges directed towards the 0 and 1 child nodes respectively. Generate a random number $r \in [0, 1]$. If $r \leq |p_0|^2$ then visit the 0 child node, else visit the 1 child node. The labels of the nodes visited is the measured state. This algorithm will reproduce the probability distribution of the qubit state and is in $O(n)$ where n is the number of qubits. QuNets implement measurements in linear time.

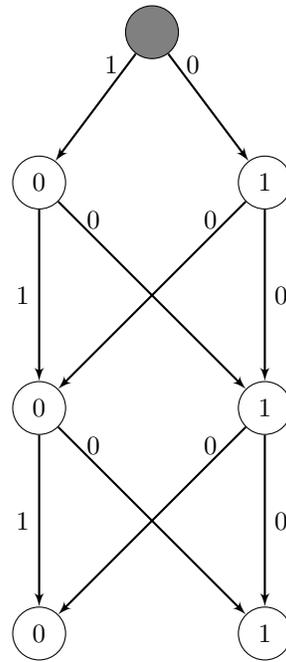


Figure 3: shows QuDot Net of $|g\rangle$

3 Single Qubit Gates

3.1 Quantum gates as edge weight transformations

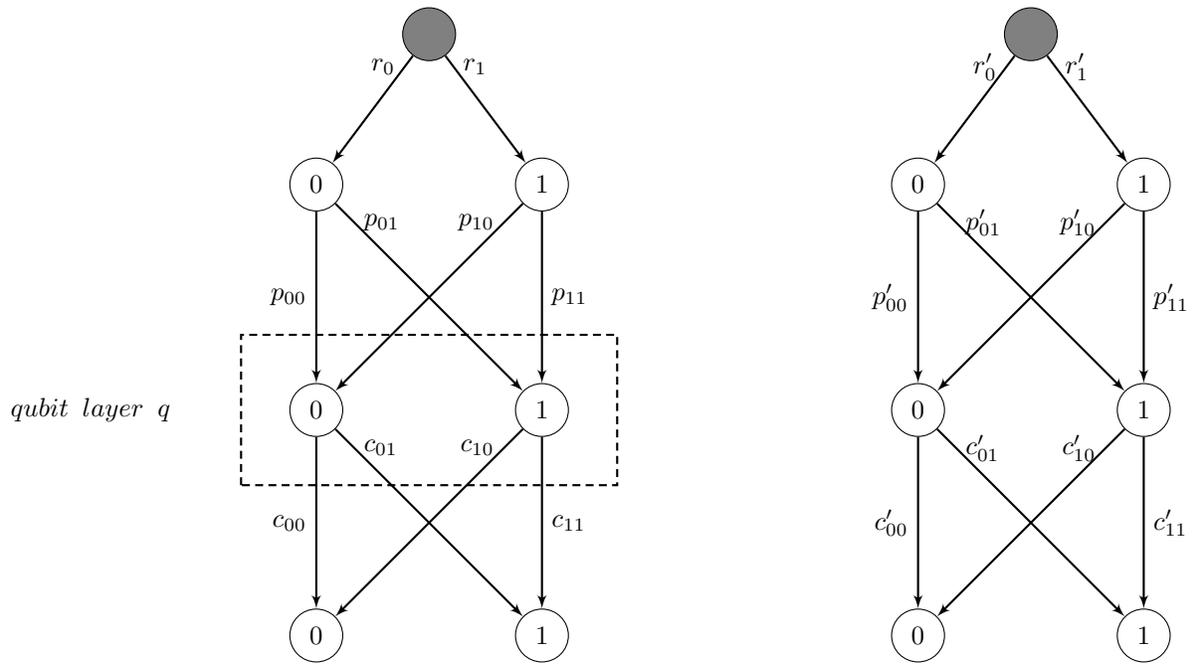
QuNets allow for the efficient implementation of some single qubit gates by treating them as operations on the edge weights. The unitary transformations that the quantum circuit model uses as quantum gates are replaced with transformations of edge weights in a qubit layer. QuNet implementations of single qubit gates define specific rules on how to transform parent and child edge weights of sibling nodes in a qubit layer as outlined in figure 4. The X , H , M and $R(k)$ single qubit gates are described in detail, however, many variations and alterations to the following details may be used to create additional single qubit gates. We implement our methods for the gates we define using the Java programming language and show results for the scalability of QuNet defined quantum gates. A clear linear scale is shown for all defined gates.

Note that special care must be taken for the first qubit because the parent node for the first qubit is the *root* node which is not another qubit node. The edges of the root node are labeled r_0 and r_1 for the edges directed towards the zero node and one node respectively. We show a separate description of the first qubit in our gates below for added clarity.

3.2 X: NOT gate

To apply the X gate to qubit q in a QuDot Net:

1. go to qubit layer q

Figure 4: single qubit gate applied on qubit q

2. swap child edge weights of siblings in qubit layer q :

$$\begin{aligned} c'_{00} &= c_{10} \\ c'_{01} &= c_{11} \\ c'_{10} &= c_{00} \\ c'_{11} &= c_{01} \end{aligned}$$

3. if $q = 1$ (root node parent):

$$\begin{aligned} r'_0 &= r_1 \\ r'_1 &= r_0 \end{aligned}$$

4. if $q \neq 1$ swap parent edge weights of siblings in qubit layer q :

$$\begin{aligned} p'_{00} &= p_{01} \\ p'_{10} &= p_{11} \\ p'_{01} &= p_{00} \\ p'_{11} &= p_{10} \end{aligned}$$

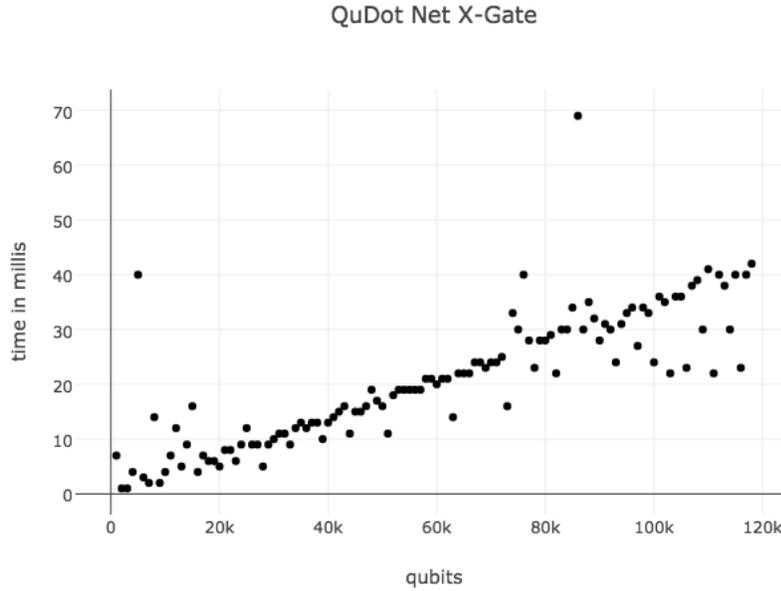


Figure 5: shows linear scaling of X gate in a QuDot Net up to 118,000 qubits

3.3 H: Haddamard gate

To apply the H gate to qubit q in a QuDot Net:

1. go to qubit layer q
2. if $q = 1$ (root node parent):

$$r'_0 = r_0 \left(\frac{1}{\sqrt{2}} \right) + r_1 \left(\frac{1}{\sqrt{2}} \right)$$

$$r'_1 = r_0 \left(\frac{1}{\sqrt{2}} \right) + r_1 \left(\frac{-1}{\sqrt{2}} \right)$$

3. if $q \neq 1$ interfere parent edges in qubit layer q :

$$p'_{00} = p_{00} \left(\frac{1}{\sqrt{2}} \right) + p_{01} \left(\frac{1}{\sqrt{2}} \right)$$

$$p'_{10} = p_{10} \left(\frac{1}{\sqrt{2}} \right) + p_{11} \left(\frac{1}{\sqrt{2}} \right)$$

$$p'_{01} = p_{00} \left(\frac{1}{\sqrt{2}} \right) + p_{01} \left(\frac{-1}{\sqrt{2}} \right)$$

$$p'_{11} = p_{10} \left(\frac{1}{\sqrt{2}} \right) + p_{11} \left(\frac{-1}{\sqrt{2}} \right)$$

4. copy nonzero child edges:

- if $c_{00} = 0$ and $c_{10} \neq 0$ or $c_{00} \neq 0$ and $c_{10} = 0$ then set $c_{00} = c_{10}$

3.3 H : Haddamard gate

- if $c_{01} = 0$ and $c_{11} \neq 0$ or $c_{01} \neq 0$ and $c_{11} = 0$ then set $c_{01} = c_{11}$

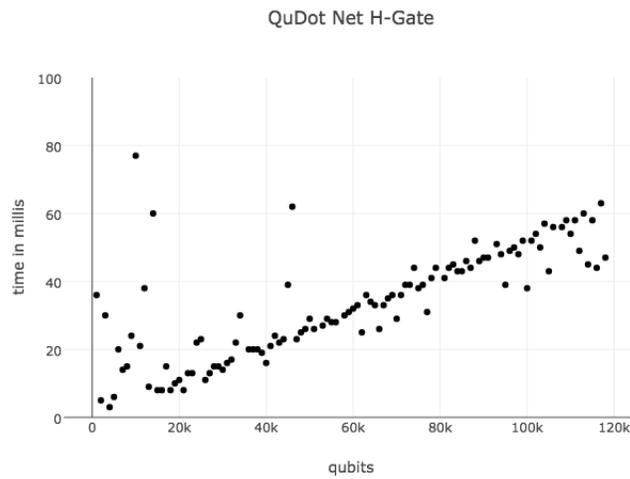


Figure 6: shows linear scaling of H gate in a QuDot Net up to 118,000 qubits

3.4 $R(k)$: phase gates

$R(k)$ gates are the family of phase gates. For each $k \geq 0$ we have a distinct gate $R(k)$. The following method applies to the family of phase gates:

$$R(k) = \begin{bmatrix} 1 & 0 \\ 0 & \phi(k) \end{bmatrix}$$

where $\phi(k)$ is the *phase* defined as

$$\phi(k) = e^{\frac{2\pi i}{2^k}}$$

To apply the $R(k)$ gate to qubit number q in a QuDot Net:

1. go to qubit layer q
2. if $q = 1$ (root node parent):

$$r'_1 = r_1 * \phi(k)$$

3. if $q \neq 1$ apply the phase to the parent edges of the 1 node:

$$p'_{01} = p_{01} * \phi(k)$$

$$p'_{11} = p_{11} * \phi(k)$$

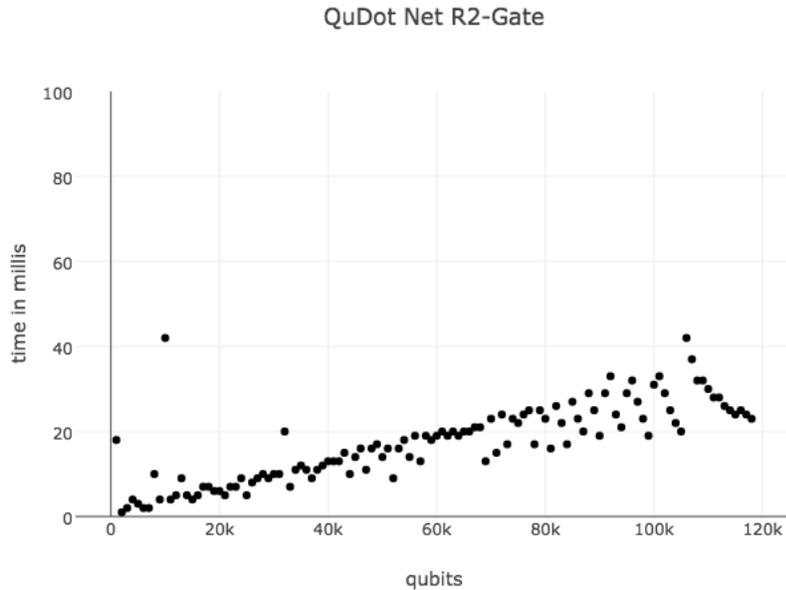


Figure 7: shows linear scaling of $R(2)$ gate in a QuDot Net up to 118,000 qubits

3.5 Accuracy of Single Qubit Gates

We tested the accuracy of the single qubit gates X , H and $R(k)$ by directly comparing 10 qubit states to a matrix mechanics emulation. We use the open source Python library *QuDotPy* for the matrix mechanics emulation. Let ρ be the density matrix of the QuNet quantum state and σ be the density matrix of the exact state obtained via matrix mechanics emulation. We use the Fidelity Measure to compare the output:

$$F(\rho, \sigma) = \text{Tr} \sqrt{\rho^{1/2} \sigma \rho^{1/2}}$$

Our testing process starts in the ground state and applies a gate to: entire state, all even qubits, subsets of even qubits, odd qubits, subset of odd qubits; finally, repeat with a state in a complete superposition. For each test we run the operation through both QuNets and a classical emulation and compare the results to obtain the fidelity. We use the Worst Fidelity measurement as our fidelity for a gate. Our Fidelity results are:

1. X Gate: 0.9999999999999986
2. H Gate: 0.9999999999999730
3. $R(k)$ Gates: 0.999998801650631

3.6 M: measurement gate

QuNets allow for the efficient implementation of the single qubit measurement gate M . The single qubit measurement gate M assigns a definite value to the qubit q of 0 or 1. To apply the single qubit measurement gate M to the qubit q :

1. go to qubit layer q
2. if the 0 node is the only node with parent edges that have nonzero weights then $q = 0$
3. if the 1 node is the only node with parent edges that have nonzero weights then $q = 1$
4. if both the 0 and 1 nodes have parent edges with nonzero weights then calculate the probability of the 0 node, defined as $P(0)$, then generate a random number between 0 and 1 defined as r . If $r \leq P(0)$ then $q = 0$, the 0 node is activated and the 1 node is deactivated. If $r > P(0)$ then $q = 1$, the 1 node is activated and the 0 node is deactivated.
 - (a) To calculate $P(0)$: the 0 node has at most two nonzero parent edges. Let w_1, w_2 be the edge weights of those two parents. Let c be the number of edge weights that are nonzero. Then
$$P(0) = \frac{|w_1 + w_2|^2}{c^2}$$
 - (b) To deactivate a node: set the weight of all parent edges to 0
 - (c) To activate a node: if the node has 2 nonzero parent edges then multiply the parent edges by $\sqrt{2}$ otherwise leave edge weights the same

4 Quantum Circuits

QūNets are able to efficiently implement quantum circuits composed of single qubit gates. We always begin in the ground state $|000\dots 0\rangle$ and apply quantum gates one at a time to the QūNet as they appear in our quantum circuits. Then we traverse the QūNet using the traversal algorithm and that traversal is our state measurement. We illustrate the procedure with a simple example. Consider the quantum circuit in figure 8:

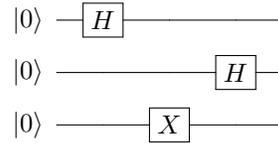


Figure 8: shows a simple quantum circuit used to explain QūDot Nets

We initialize our QūNet in the ground state $|q\rangle$ shown in figure 9. Note that edges having a weight of 0 have been omitted from the drawing of the QūNet to make the graph easier to follow for the reader.

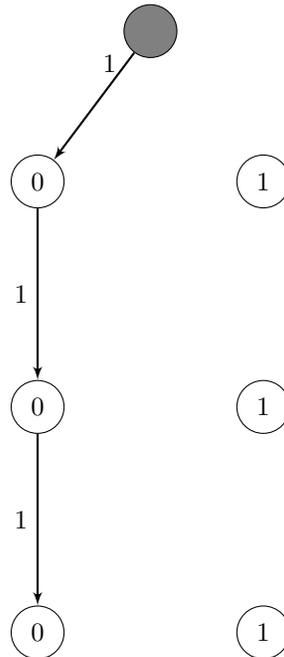


Figure 9: shows QūDot Net of $|q\rangle = |000\rangle$

Apply H to qubit 1:

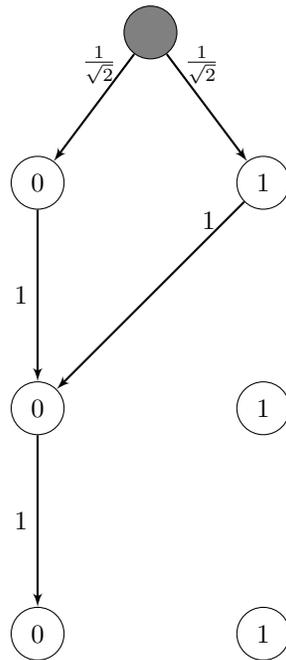


Figure 10: shows QuDot Net of $|q\rangle = \frac{1}{\sqrt{2}} (|000\rangle + |100\rangle)$

Apply X to qubit 3:

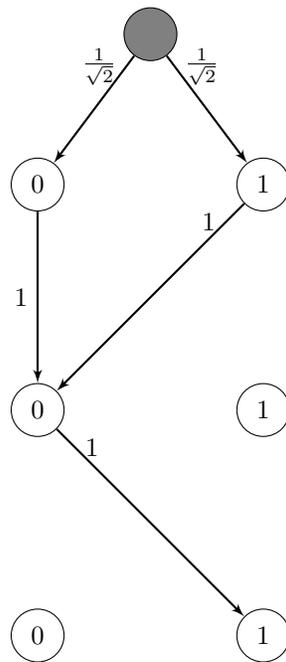


Figure 11: shows QuDot Net of $|q\rangle = \frac{1}{\sqrt{2}} (|001\rangle + |101\rangle)$

Apply H to qubit 2:

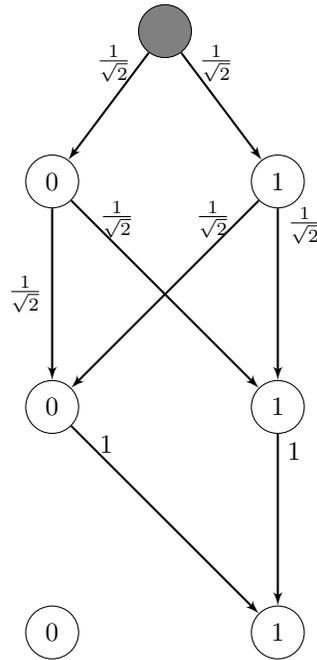


Figure 12: shows QuDot Net of $|q\rangle = \frac{1}{2}(|001\rangle + |011\rangle + |101\rangle + |111\rangle)$

If we now run our QuNet traversal algorithm we will perform a measurement thus concluding the implementation of the quantum circuit in figure 8. If we perform the QuNet traversal algorithm many times we will obtain different traversals corresponding to the different possible measurements of the state $|q\rangle$. The probability distribution we obtain from performing many traversals of the QuNet in figure 12 matches the probability distribution of $|q\rangle$, for example, we will obtain the traversal 011 with a probability of $\frac{1}{4}$.

5 Semi-Quantum Computation

5.1 semi-quantum control gates

QuNets can be used to efficiently implement semi-quantum circuits on classical hardware. A semi-quantum circuit is defined by how a control- U operation (where U is a quantum gate) is handled. If your method measures the control qubit and uses the result of the measurement to decide whether the U gate should be applied to the target qubit; then it is called a semi-quantum gate and is an application of the *Principle of Deferred Measurement* depicted in figure 13.

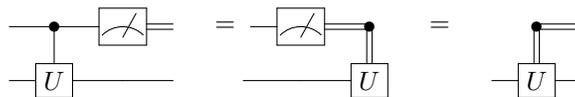


Figure 13: Principle of Deferred Measurement and semi-quantum control- U gate

After a measurement is performed on the control qubit then it collapses to a definite value and further interference cannot occur. The control qubit becomes a classical bit after measurement, therefore, the only operations allowed after a semi-quantum control gate on the control qubit are other measurements or

controls. Circuits that are composed entirely of single qubit gates, semi-quantum control gates and treat the control qubit classically after a control are called *terminal* or *semi-quantum* circuits. One such circuit is the Quantum Fourier Transform (QFT) which we implement using QuNets. Semi-quantum control gates are implemented with QuNets by measuring the control qubit using the M gate and applying the U single qubit gate on the target qubit if the measurement result is 1.

5.2 SWAP gate

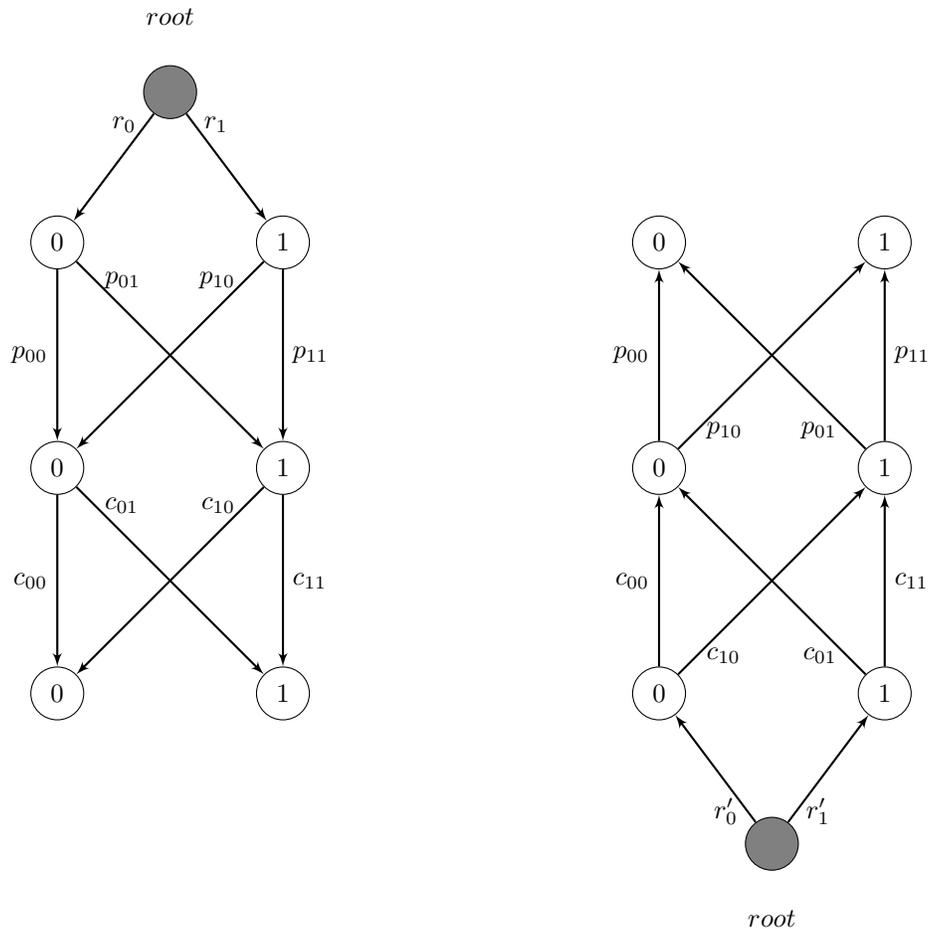


Figure 14: shows QuDot Net before and after a SWAP operation

The last quantum gate needed to implement the QFT with QuNets is the SWAP gate. QuNets can efficiently implement the n -qubit SWAP gate where n is the number of qubits in the quantum state. The n -qubit SWAP gate is not a collection of two qubit SWAP gates. The n -qubit SWAP gate operates on the entire state. QuNets efficiently implement the SWAP gate by reversing the direction of all the edges in the QuNet and pointing the root node edges to the n^{th} qubit layer. This is best illustrated in a diagram, see Figure 14

5.3 Quantum Fourier Transform

5.3.1 Scaling

The QFT (and its inverse) is one of the most well studied algorithms in quantum computation. If the QFT is the last element of a quantum circuit then it is known as the *terminal* QFT or *semi-quantum* QFT and can be classically implemented [4]. QuNets can also efficiently implement the semi-quantum QFT. We first re-write the QFT circuit in its semi-quantum form. Figure 15 shows the three qubit QFT. We use three qubits for the illustration

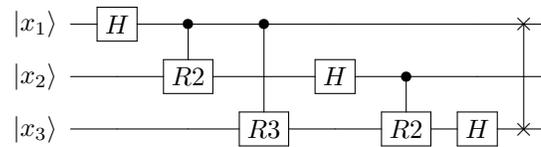


Figure 15: shows a 3 qubit Quantum Fourier Transform

We now re-write the three qubit QFT in its semi-quantum form depicted in figure 16:

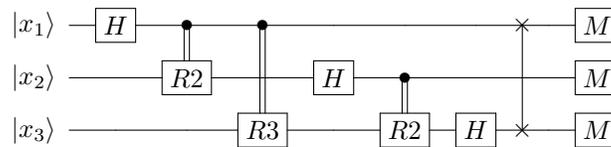


Figure 16: shows a 3 qubit Semi-Quantum Fourier Transform

Similarly, we rewrite the IQFT in its semi-quantum form:

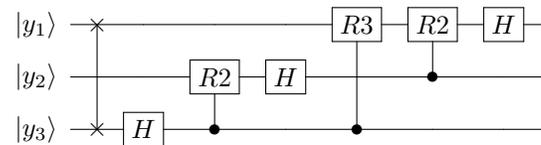


Figure 17: shows a 3 qubit inverse Quantum Fourier Transform

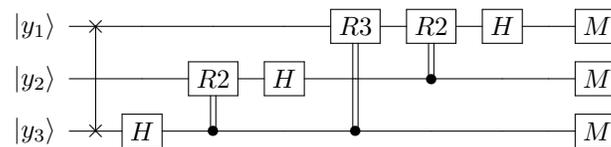


Figure 18: shows a 3 qubit Inverse Semi-Quantum Fourier Transform

The QFT and IQFT require n^2 number of gates where n is the number of qubits. An efficient implementation of the QFT and IQFT would require to scale as $O(n^2)$. We show quadratic scaling for the QFT and its inverse in our QuNet implementation for 6,000 qubits.

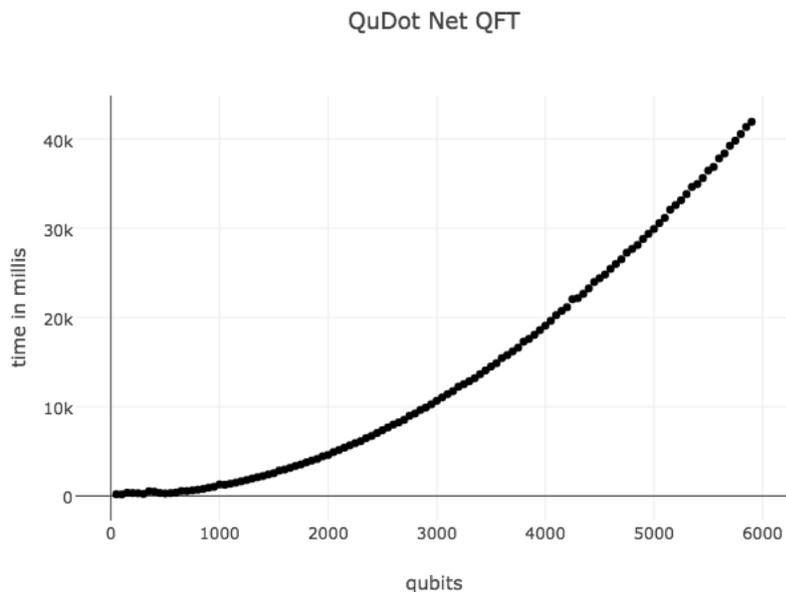


Figure 19: shows quadratic scaling in QuDot Net execution of Quantum Fourier Transform

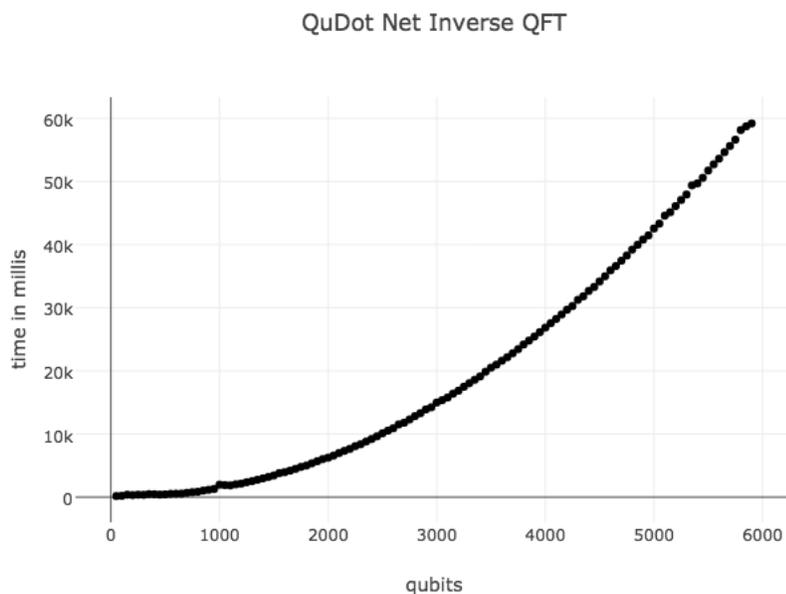


Figure 20: shows quadratic scaling in QuDot Net execution of Inverse Quantum Fourier Transform

5.3.2 Accuracy

We test the accuracy of the QuNet implementation of the QFT by comparing to exact results against small four qubit test cases. We define six quantum states and show their expected probability distribution after the QFT is performed. We then run the QuNet implementation of the QFT on an ensemble of each state

and perform a measurement. As the ensemble size increases for each state we observe that the expected probability distribution matches the observed probability distribution thus confirming the accuracy of the QuNet QFT implementation shown in figure 21.

$|qft12\rangle$ is obtained by applying H to qubits 1 and 2. $|qft24\rangle$ is obtained by applying H to qubits 2 and 4. The same mechanism is used to generate the other states. Expected probability distributions are shown in the Appendix. Figure 21 shows percent error of expected probability distribution vs measured probability distribution as ensemble size increases to 500,000 states.

1. $|qft12\rangle = 0.5 |0000\rangle + 0.5 |0100\rangle + 0.5 |1000\rangle + 0.5 |1100\rangle$
2. $|qft13\rangle = 0.5 |0000\rangle + 0.5 |0010\rangle + 0.5 |1000\rangle + 0.5 |1010\rangle$
3. $|qft14\rangle = 0.5 |0000\rangle + 0.5 |0001\rangle + 0.5 |1000\rangle + 0.5 |1001\rangle$
4. $|qft23\rangle = 0.5 |0000\rangle + 0.5 |0010\rangle + 0.5 |0100\rangle + 0.5 |0110\rangle$
5. $|qft24\rangle = 0.5 |0000\rangle + 0.5 |0001\rangle + 0.5 |0100\rangle + 0.5 |0101\rangle$
6. $|qft34\rangle = 0.5 |0000\rangle + 0.5 |0001\rangle + 0.5 |0010\rangle + 0.5 |0011\rangle$

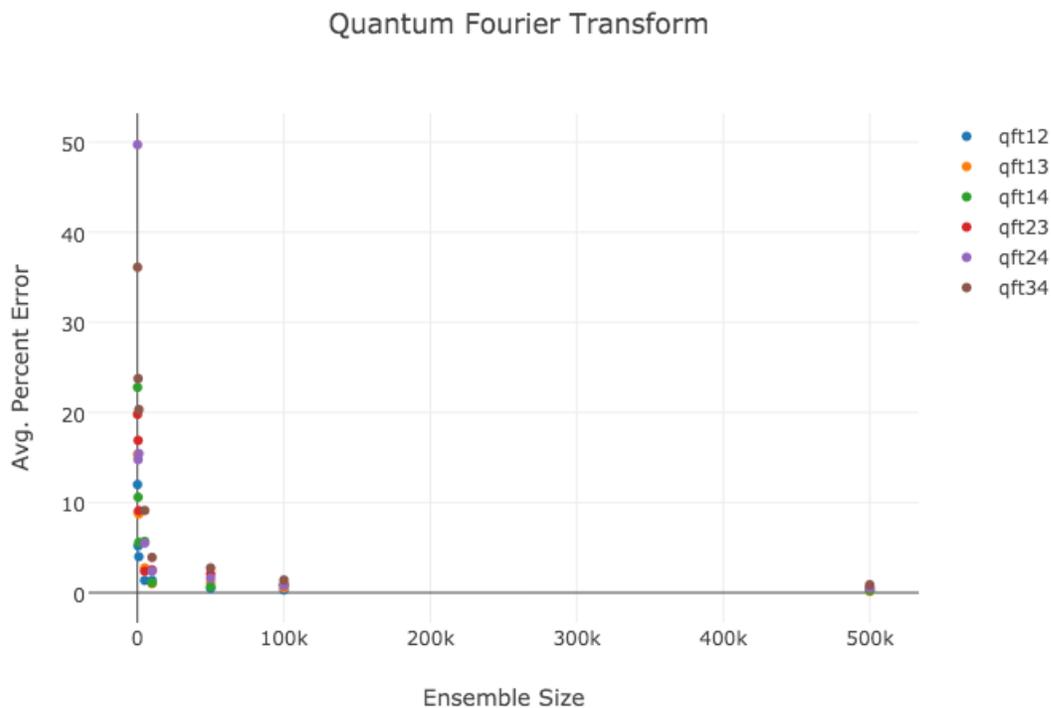


Figure 21: shows exponential decrease in percent error as ensemble size grows

6 Conclusive Remarks and Future Work

We saw how the circuit model of quantum computation can be mapped to a special type of multi-digraph inspired by Bayesian Networks called a QuDot Net. With a QuDot Net, possible measurements are possible traversals where the probability amplitude is the product of each edge weight in the traversal. Single qubit gates can be implemented by providing rules for the manipulation of edge weights in a qubit layer. Two qubit control gates can be implemented by splitting the computational path making the QuDot Net a multi-digraph. We saw that semi-quantum computation can be efficiently implemented using QuDot Nets and we provided results for the efficient implementation of the QFT and Inverse QFT.

What makes the quantum model of computation exponentially more powerful than the classical model of computation? This is still an open area of research. QuDot Nets show us that it is control gates that provide us with more power. A control gate splits the computational path and nature somehow keeps track of all the different edges then combines them in the correct way. The most interesting question regarding QuDot Nets is “Can we control the number of computational paths so they do not grow exponentially?”. If we can then quantum computation can be efficiently implemented using classical hardware. If we cannot, can we at least approximate the answer using well know approximation techniques common in other NP-complete graph theoretic problems such as traveling salesmen and the knapsack problems? Even if there are errors in the approximations we still may be able to beat quantum hardware since quantum hardware also has errors but further work is needed to explore this possibility. Even if classical computers are exponentially slower than quantum computers for some problems, classical simulation of quantum circuits will allow us to design better quantum algorithms by exploring the region where quantum supremacy begins.

7 Appendix

Table of expected probability distributions after applying QFT to test states used in Figure 21.

input state	possible measurement	measurement probability
$ qft12\rangle$	$ 0000\rangle$	0.2499999999999972
	$ 0100\rangle$	0.2499999999999972
	$ 1000\rangle$	0.2499999999999972
	$ 1100\rangle$	0.2499999999999972
$ qft13\rangle$	$ 0000\rangle$	0.2499999999999972
	$ 0010\rangle$	0.1249999999999988
	$ 0110\rangle$	0.1249999999999985
	$ 1000\rangle$	0.2499999999999972
	$ 1010\rangle$	0.1249999999999988
	$ 1110\rangle$	0.1249999999999985
$ qft14\rangle$	$ 0000\rangle$	0.2499999999999972
	$ 0010\rangle$	0.21338834764831818
	$ 0100\rangle$	0.1249999999999988
	$ 0110\rangle$	0.03661165235168154
	$ 1010\rangle$	0.03661165235168150
	$ 1100\rangle$	0.1249999999999985
	$ 1110\rangle$	0.21338834764831816
$ qft23\rangle$	$ 0000\rangle$	0.2499999999999972
	$ 0001\rangle$	0.10669417382415913
	$ 0011\rangle$	0.01830582617584076
	$ 0101\rangle$	0.01830582617584075
	$ 0111\rangle$	0.10669417382415908
	$ 1000\rangle$	0.2499999999999972
	$ 1001\rangle$	0.10669417382415913
	$ 1011\rangle$	0.01830582617584076
	$ 1101\rangle$	0.01830582617584075
	$ 1111\rangle$	0.10669417382415908
$ qft24\rangle$	$ 0000\rangle$	0.2499999999999972
	$ 0001\rangle$	0.12024247078195528
	$ 0011\rangle$	0.08641771452281801
	$ 0100\rangle$	0.1249999999999988
	$ 0101\rangle$	0.03858228547718185
	$ 0111\rangle$	0.00475752921804457
	$ 1001\rangle$	0.00475752921804457
	$ 1011\rangle$	0.03858228547718183
	$ 1100\rangle$	0.1249999999999985
	$ 1101\rangle$	0.08641771452281801
	$ 1111\rangle$	0.12024247078195528

input state	possible measurement	measurement probability
$ qft34\rangle$	$ 0000\rangle$	0.24999999999999972
	$ 0001\rangle$	0.20526673725850120
	$ 0010\rangle$	0.10669417382415913
	$ 0011\rangle$	0.02531116256909020
	$ 0101\rangle$	0.01130048978259131
	$ 0110\rangle$	0.01830582617584077
	$ 0111\rangle$	0.00812161038981702
	$ 1001\rangle$	0.00812161038981702
	$ 1010\rangle$	0.01830582617584075
	$ 1011\rangle$	0.01130048978259130
	$ 1101\rangle$	0.02531116256909020
	$ 1110\rangle$	0.10669417382415908
	$ 1111\rangle$	0.20526673725850114

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